

DENSITY OF KINKS JUST AFTER A QUENCH IN AN OVERDAMPED SYSTEM

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(February 19, 1998)

A quench in an overdamped one dimensional ϕ^4 model is studied by analytical and numerical methods. For an infinite system or a finite system with free boundary conditions, the density of kinks after the transition is proportional to the eighth root of the rate of the quench. For a system with periodic boundary conditions, it is proportional to the fourth root of the rate. The critical exponent predicted in Zurek scenario is put in question.

Topological defects play a prominent role in many condensed matter systems, see e.g. [1] for a review. It was also suggested that they were an important ingredient of the early universe [2]. Topological defects can be generated in large numbers during a second order phase transition. The dynamics of such a transition to a symmetry broken phase has been an object of much recent attention because of its importance in the cosmological context [3] and in condensed matter systems [4].

An early estimate of defect density after a quench was given by Kibble [5]. In his theory, the speed of light is a dominant factor which determines the size of correlated domains. An alternative scenario was put forward by Zurek [6], who emphasized the importance of the nonequilibrium dynamics of the order parameter. Recent experiments point to the latter theory. The principal prediction of Ref. [6] is the scaling law. He is able to predict by very general arguments an exponent in a power law dependence of the density of topological defects on a time of the quench in a variety of systems. The numerical experiment [7] seems to confirm this prediction.

It is normally assumed that close to the phase transition the dynamics of the system is overdamped. We follow this line of argument to its logical limit and concentrate from the very beginning on a purely diffusive dynamics. The model we study is the ϕ^4 theory in one spatial dimension. Although there is no proper phase transition in such a one dimensional model there is a long range order in its symmetry broken phase. The real field ϕ falls into one of the two minima of the Ginzburg-Landau potential at $\phi = +1$ or $\phi = -1$. The domain walls separating these two different vacua are the topological solitons or the kinks. We develop a novel method of how to predict the density of kinks after a quench. It

turns out that the density is proportional to the eighth root of the quench rate in contradiction with Zurek's prediction of the fourth root. We point to an implicit assumption in Zurek scenario, which lays at the bottom of this discrepancy.

Let us consider a one dimensional overdamped ϕ^4 model

$$\dot{\phi}(t, x) = \phi''(t, x) - 2a(t)\phi(t, x) - 2\phi^3(t, x) + \eta(t, x) , \quad (1)$$

where $\dot{} \equiv \partial_t$, $' \equiv \partial_x$. $\phi(t, x)$ is a real order parameter. $\eta(t, x)$ is a Gaussian noise of temperature T with correlations

$$\begin{aligned} \langle \eta(t, x) \rangle &= 0 , \\ \langle \eta(t_1, x_1)\eta(t_2, x_2) \rangle &= 2T\delta(t_1 - t_2)\delta(x_1 - x_2) . \end{aligned} \quad (2)$$

The coefficient $a(t)$ is time dependent. We consider a linear quench

$$a(t) = \begin{cases} A & , \text{ if } t < 0 \\ A(1 - \frac{t}{\tau}) & , \text{ if } 0 < t < \tau \frac{A+1}{A} \\ -1 & , \text{ if } \tau \frac{A+1}{A} < t \end{cases} \quad (3)$$

Before the quench, for $t < 0$, the system is in a symmetric phase ($A > 0$), during the quench, at $t = \tau$, it undergoes a transition from the symmetric phase $a(t < \tau) > 0$ to a broken symmetry phase $a(t > \tau) < 0$. Finally it settles down at $a(t) = -1$.

As long as the system is still in the symmetric phase ($t < \tau$), the field ϕ can be regarded as a small fluctuation around its symmetric ground state $\phi(t, x) = 0$. It is justified to neglect the cubic term on the RHS of Eq.(1). The field can be written as a Fourier transform,

$$\phi(t, x) = \int_{-\infty}^{+\infty} dk \tilde{\phi}(t, k) e^{ikx} . \quad (4)$$

The Fourier transform of linearized Eq.(1) is

$$\dot{\tilde{\phi}}(t, k) = -[2a(t) + k^2]\tilde{\phi}(t, k) + \tilde{\eta}(t, k) . \quad (5)$$

The Fourier transformed noises have correlations

$$\begin{aligned} \langle \tilde{\eta}(t, k) \rangle &= 0 , \\ \langle \tilde{\eta}^*(t_1, k_1)\tilde{\eta}(t_2, k_2) \rangle &= \frac{T}{\pi}\delta(t_1 - t_2)\delta(k_1 - k_2) . \end{aligned} \quad (6)$$

The formal solution of Eq.(5) at the time τ is

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$$\tilde{\phi}(\tau, k) = \int_{-\infty}^{\tau} dt_1 \exp\left\{-\int_{t_1}^{\tau} dt_2 [2a(t_2) + k^2]\right\} \tilde{\eta}(t, k) . \quad (7)$$

This solution, the correlations (6) and the explicit form of $a(t)$, see Eq.(3), give the correlation function at $t = \tau$

$$\begin{aligned} \langle \tilde{\phi}^*(t_1, k_1) \tilde{\phi}(t_2, k_2) \rangle &= \frac{T}{\pi} \delta(k_1 - k_2) G(\tau, A, k) \\ G(\tau, A, k) &= \frac{e^{-2k^2\tau}}{4A + 2k^2} + \\ &\sqrt{\frac{\pi\tau}{8A}} \left[\text{Erf}\left(\sqrt{\frac{\tau}{2A}} (2A + k^2)\right) - \text{Erf}\left(\sqrt{\frac{\tau}{2A}} k^2\right) \right] . \quad (8) \end{aligned}$$

The linearized approximation can not be used in the broken symmetry phase. Nevertheless we would like to use the correlations at $t = \tau$ to predict the density of topological defects after the quench. We have to make contact between the formalism developed for the symmetric phase and the one which is relevant to the symmetry broken phase.

$a(t) = -1$ in the symmetry broken phase. If the noise is neglected, then Eq.(1) has a static kink solution

$$\phi(t, x) = \tanh(x - z) \equiv K(x - z) \quad (9)$$

with an arbitrary kink position z . The field which is going to relax to this equilibrium configuration (9) can be expressed as a sum of the kink solution and of eigenfunctions u_α of the fluctuation operator around the kink numbered by a discrete/continuous index α ,

$$\phi(t, x) = K(x - z) + \sum_{\alpha} \Phi_{\alpha}(t) u_{\alpha}(x - z) . \quad (10)$$

The eigenstates have positive eigenvalues and their amplitudes $\Phi_{\alpha}(t)$ decay exponentially with time. The "kink transform" of the field (10)

$$I(t, x) = \int_{-\infty}^{+\infty} dy K'(y) \phi(t, x + y) \quad (11)$$

vanishes for $x = z$ because $K'(x)$ is orthogonal to $K(x)$ and to $u_{\alpha}(x)$'s. The zero of the kink transform of the initial field is the equilibrium position of the kink. The same transform (11) can be used to predict positions of antikinks. This method has been introduced and extensively tested numerically in [8]. To predict the positions of kinks after the quench one must find the kink transform $I(\tau, x)$ of the initial field $\phi(\tau, x)$ and locate the zeros of $I(\tau, x)$.

We do not need any detailed knowledge about the actual positions of kinks. All we need to know is their average density. The density of kinks n can be expressed by the density of zeros of the kink transform $I(\tau, x)$

$$n = \frac{\langle N \rangle}{2L} = \frac{1}{2L} \int_{-L}^L dx \langle |I'(\tau, x)| \delta[I(\tau, x)] \rangle . \quad (12)$$

L is the length of the considered interval. Due to translational invariance

$$n = \langle |I'(\tau, 0)| \delta[I(\tau, 0)] \rangle . \quad (13)$$

Thanks to the same symmetry $\langle I(\tau, x) I'(\tau, x) \rangle = 0$ for any x , so that

$$n = \langle |I'(\tau, 0)| \rangle \langle \delta[I(\tau, 0)] \rangle . \quad (14)$$

The identity $|I'| = I' \text{Sign}(I')$ and the Fourier transforms of the δ and Sign functions help to derive

$$n = \frac{\pi}{2} \sqrt{\frac{\langle I'^2(\tau, 0) \rangle}{\langle I^2(\tau, 0) \rangle}} . \quad (15)$$

The higher is the ratio of the variation of I to its average magnitude, the higher is the density of kinks. The higher is this ratio, the more likely is the function I to cross zero.

One can express the kink transform with the Fourier transform of $\phi(\tau, x)$ (4) as

$$\begin{aligned} I(\tau, 0) &= \int_{-\infty}^{+\infty} dk U(k) \tilde{\phi}(\tau, k) , \\ I'(\tau, 0) &= i \int_{-\infty}^{+\infty} dk k U(k) \tilde{\phi}(\tau, k) , \\ U(k) &= \frac{\pi k}{\sinh(\pi k/2)} . \quad (16) \end{aligned}$$

$U(k)$ is a Fourier transform of $K'(x)$. The $I - I$ correlations in Eq.(15) can be expressed by the correlations (8), so that finally we obtain

$$n = \frac{\pi}{2} \sqrt{\frac{\int dk k^2 U^2(k) G(\tau, A, k)}{\int dk U^2(k) G(\tau, A, k)}} . \quad (17)$$

The kink size provides an ultraviolet cut-off through the function $U^2(k)$. Without any UV cut-off the expression (17) would be infinite for any A and τ . In the symmetric phase the noise driven order parameter $\phi(t, x)$ has an infinite number of zeros.

To extract the scaling law, we must find the asymptote of n for large τ . The asymptote of the correlation function G is

$$G(\tau, A, k) \approx \sqrt{\frac{\tau}{2A}} e^{2A\tau + \frac{k^4\tau}{2A}} \int_{\sqrt{\frac{\tau}{2A}}k^2}^{+\infty} dx e^{-x^2} . \quad (18)$$

We can make further approximation under the integral in the numerator of (17)

$$\begin{aligned} \int_{-\infty}^{+\infty} dk k^2 U^2(k) G(\tau, A, k) &\approx \\ \frac{1}{2} e^{2A\tau} \int_{-\infty}^{+\infty} dk U^2(k) &= \frac{4\pi}{3} e^{2A\tau} . \quad (19) \end{aligned}$$

We introduce $p = k (\frac{\tau}{2A})^{1/4}$ in the denominator integral to obtain the asymptote

$$\begin{aligned}
& \int_{-\infty}^{+\infty} dk U^2(k) G(\tau, A, k) \approx \\
& \left(\frac{\tau}{2A}\right)^{\frac{1}{4}} e^{2A\tau} \int_{-\infty}^{+\infty} dp U^2\left[\frac{p}{\left(\frac{\tau}{2A}\right)^{\frac{1}{4}}}\right] e^{p^4} \int_{p^2}^{+\infty} dx e^{-x^2} \approx \\
& \left(\frac{\tau}{2A}\right)^{\frac{1}{4}} e^{2A\tau} U^2(0) \int_{-\infty}^{+\infty} dp e^{p^4} \int_{p^2}^{+\infty} dx e^{-x^2}. \quad (20)
\end{aligned}$$

After the numerical coefficients are worked out one obtains

$$n \approx 1.16 \left(\frac{A}{\tau}\right)^{\frac{1}{8}}. \quad (21)$$

The density of kinks is proportional to the eighth root of the transition rate. The asymptote is achieved for $\tau \gg 2A$. The critical exponent does not depend on the actual choice of the UV cut-off provided by the function $U^2(k)$. In a lattice theory, regularized by its lattice constant, one would get the $1/8$ critical exponent for the density of zeros of the order parameter $\phi(\tau, x)$ itself.

The last prediction must be contrasted with a corresponding result for a model with periodic boundary conditions. The periodicity $\phi(t, L) = \phi(t, 0)$ implies that momentum is quantized $k_m = (\frac{2\pi}{L})m$ with integer m . The integrals over k in (17) are replaced by summations over m and all appearances of k are replaced by k_m . The denominator sum is dominated for large τ by the $m = 0$ mode contribution,

$$\begin{aligned}
& \sum_{m=-\infty}^{+\infty} U^2(k_m) G(\tau, A, k_m) \approx \\
& \sum_{m=-\infty}^{+\infty} U^2(k_m) \sqrt{\frac{\tau}{2A}} e^{2A\tau + \frac{k_m^4}{2A}\tau} \int_{\sqrt{\frac{\tau}{2A}}k_m^2}^{+\infty} dx e^{-x^2} \approx \\
& \sqrt{\frac{\tau}{2A}} e^{2A\tau} U^2(0) \int_0^{+\infty} e^{-x^2}. \quad (22)
\end{aligned}$$

The numerator sum is approximated in a similar way as the numerator integral for the infinite system,

$$\begin{aligned}
& \sum_{m=-\infty}^{+\infty} k_m^2 U^2(k_m) G(\tau, A, k_m) \approx \\
& \sum_{m=-\infty}^{+\infty} k_m^2 U^2(k_m) \sqrt{\frac{\tau}{2A}} e^{2A\tau + \frac{k_m^4}{2A}\tau} \int_{\sqrt{\frac{\tau}{2A}}k_m^2}^{+\infty} dx e^{-x^2} \approx \\
& \frac{1}{2} e^{2A\tau} \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} U^2(k_m) \quad (23)
\end{aligned}$$

Putting these two asymptotes together we get

$$n \approx \sqrt{\frac{\pi^{\frac{3}{2}} \sum_m U^2(k_m)}{8}} \left(\frac{A}{\tau}\right)^{\frac{1}{4}}. \quad (24)$$

If the periodic boundary conditions are imposed, the density of kinks is proportional to the fourth root of the

transition rate. In fact this conclusion is the same for any fixed boundary conditions which quantize k . The asymptote is achieved for $\sqrt{\frac{\tau}{2A}} k_1^2 = \sqrt{\frac{\tau}{2A}} \frac{(2\pi)^2}{L^2} \gg 1$. We expect for a large but finite system, that the density scales like the eighth root of the quench rate for moderately large τ but for very large τ it scales like the fourth root. Transition between these two regimes takes place at a τ which grows like the fourth power of the system size L . The thermodynamic limit and the large quench time limit do not commute. Let us note in passing that a similar dependence of critical exponent on boundary conditions takes place in the diffusion of overdamped kinks in the sine-Gordon chain [9].

The Zurek scenario as described in [6] does not refer to boundary conditions. Let us analyse our derivation from the point of view of Ref. [6]. The relaxation time of the system depends on time like $t_r = \frac{1}{2a(t)}$. The system ceases to follow the changes in $a(t)$ at the instant \hat{t} , when its relaxation time becomes greater than the time left to the transition at $t = \tau$, $(\tau - \hat{t}) = \frac{1}{2a(\hat{t})}$. For the linear quench (3) this takes place when $(\tau - \hat{t}) = \sqrt{\frac{\tau}{2A}}$. At this instant of time the correlation length is $\xi = (\frac{\tau}{2A})^{1/4}$. The density of kinks after the transition is inversely proportional to this "frozen" correlation length, so $n \sim (\frac{A}{\tau})^{1/4}$.

The weak point of this argument is that $t_r = \frac{1}{2a(t)}$ is the relaxation time but only for the $k = 0$ mode. Modes with nonzero momenta $k > 0$ have relaxation times $t_r(k) = \frac{1}{2a(t) + k^2} < t_r(0)$ and they are frozen later than the $k = 0$ mode. As we can see in Eq.(15), the density of nucleated kinks is related to the ratio of the magnitude of short wavelength modes to the magnitude of long wavelength modes. Contrary to Zurek scenario, which implicitly assumes that all the modes are frozen at the same instant of time, the $k > 0$ modes can still grow for a while even after the $k = 0$ mode was already frozen. Thus the density of kinks after quench is greater than that given by Zurek estimate. It is a matter of detailed calculations presented above to find out that the actual critical exponent is $1/8$ instead of $1/4$.

It is the point to discuss the numerical experiment of Laguna and Zurek [7]. Those authors discuss the model (1) with $\ddot{\phi}$ added on the LHS. Their numerical experiment on a lattice with periodic boundary conditions gives a critical exponent of $1/4$. We could naively argue that this is due to the periodic boundary conditions and in agreement with our theory. This is, however, not the case. The lattice [7] is large enough to render the boundary conditions irrelevant. Hence according to our theory one would expect the $1/8$ scaling. But the point is that neither Zurek estimate nor our calculations apply to this system. The damping coefficient $\eta = 1$ used in actual simulations [7] (in distinction to the theoretical discussion [7]) is in the underdamped range. At the beginning of the quench all the field oscillators with different momenta k are underdamped. The $k = 0$ would be overdamped, if $\eta > \sqrt{2}$ in units of Ref. [7]. The short

wavelength modes would require even higher η . At the beginning of the quench the second order time derivative dominates over the first order derivative. Without any reliable tool at hand we can only make a very superficial remark. Replacement of $\dot{\phi}$ by $\ddot{\phi}$ should give the replacement of τ by τ^2 in any scaling behavior. This replacement in our formula (21) leads to the critical exponent 1/4 in agreement with the numerics of Ref. [7].

We performed a numerical simulation of the quench (3) in the model (1). The total extension of our lattice was 20 units, the lattice constant 0.1 and the temperature $T = 10^{-8}$. With these parameters the periodic lattice system was far in the thermodynamic regime, the asymptote with the 1/4 scaling is expected to be achieved for $\tau \gg 10^4$. The parameter A in Eq.(3) was chosen as $A = 100$. We have measured the density of zeros of the function $\phi(\tau, x)$ for $\tau = 0.001 \dots 3.2$ units of time. The simulation was repeated 20 times for each τ . The figure shows the double logarithmic plot of the density of zeros n_z as a function of the quench time. The slope is -0.110 ± 0.005 , which is consistent with our prediction of $-\frac{1}{8} = -0.125$.

Conclusion.

The critical exponent observed in an experiment on a macroscopic system in its overdamped regime should be 1/8. One can observe the 1/4 scaling in a numerical "experiment" on a small lattice.

ACKNOWLEDGMENTS

I would like to thank Markus Büttiker for sending me a reprint of Ref. [9].

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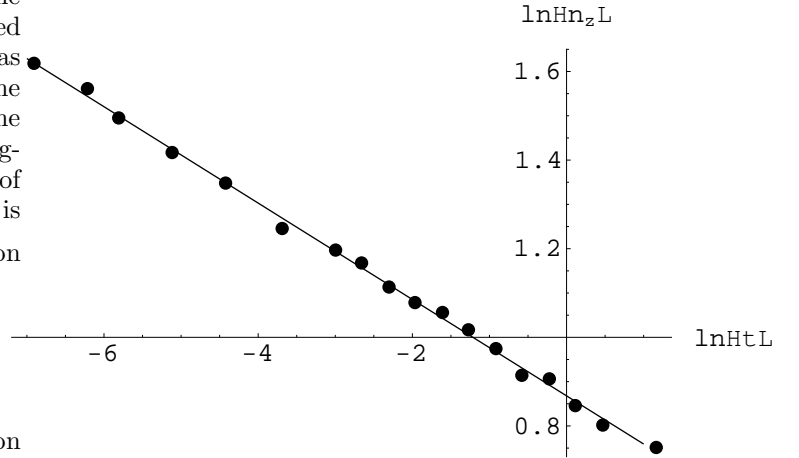


Figure caption. Logarithm of the density of zeros of the function $\phi(\tau, x)$ as a function of the logarithm of the quench time τ . The slope of the fitted line is -0.110 ± 0.005 , which is consistent with the prediction of $-\frac{1}{8} = -0.125$.

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